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Published in:
SIAM Journal on Optimization

Publication date:
2012

Document Version
Publisher's PDF, also known as Version of record

SIAM J. OPTIM.
Vol. 22, No. 2, pp. 581–595

IMPROVED LOWER BOUNDS FOR THE 2-PAGE CROSSING NUMBERS OF $K_{m,n}$ AND K_n VIA SEMIDEFINITE PROGRAMMING*

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Abstract. It has long been conjectured that the crossing numbers of the complete bipartite graph $K_{m,n}$ and of the complete graph K_n equal $Z(m,n) := \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$ and $Z(n) := \frac{1}{4} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$, respectively. In a 2-*page drawing* of a graph, the vertices are drawn on a straight line (the *spine*), and each edge is contained in one of the half-planes of the spine. The 2-*page crossing number* $\nu_2(G)$ of a graph G is the minimum number of crossings in a 2-page drawing of G. Somewhat surprisingly, there are 2-page drawings of $K_{m,n}$ (respectively, K_n) with exactly $Z(m,n)$ (respectively, $Z(n)$) crossings, thus yielding the conjectures (I) $\nu_2(K_{m,n}) \stackrel{?}{=} Z(m,n)$ and (II) $\nu_2(K_n) \stackrel{?}{=} Z(n)$. It is known that (I) holds for $\min\{m,n\} \leq 6$, and that (II) holds for $n \leq 14$. In this paper we prove that (I) holds asymptotically (that is, $\lim_{n\to\infty} \nu_2(K_{m,n})/Z(m,n) = 1$) for $m = 7$ and 8. We also prove (II) for $15 \leq n \leq 18$ and $n \in \{20, 24\}$, and establish the asymptotic estimate $\lim_{n\to\infty} \nu_2(K_n)/Z(n) \geq 0.9253$. The previous best-known lower bound involved the constant 0.8594.

Key words. 2-page crossing number, book crossing number, semidefinite programming, maximum cut, Goemans–Williamson max-cut bound

AMS subject classifications. 90C22, 90C25, 05C10, 05C62, 57M15, 68R10

DOI. 10.1137/110852206

1. Introduction. We recall that the *crossing number* $\mathrm{cr}(G)$ of a graph G is the minimum number of pairwise intersections of edges (at a point other than a vertex) in a drawing of G in the plane. Besides their natural interest in topological graph theory, crossing number problems are of interest because of their applications, most notably in VLSI design [23].

Also motivated by applications to VLSI design, Chung, Leighton, and Rosenberg [4] studied embeddings of graphs in *books*: the vertices are placed along a line (the *spine*) and the edges are placed in the *pages* of the book. In a *book drawing* (equivalently, k-*page drawing*, if the book has k pages), crossings among edges are allowed. The k-*page crossing number* $\nu_k(G)$ of a graph G is the minimum number of crossings of edges in a k-page drawing of G.

Clearly, a graph G has $\nu_1(G) = 0$ if and only if it is outerplanar. Closely related to 1-page drawings are *circular drawings*, in which the vertices are placed on a circle and all edges are drawn in its interior. It is easy to see the one-to-one correspondence between 1-page drawings and circular drawings.

In a similar vein, 2-page drawings can be alternatively modeled by drawing the vertices of the graph on a circle, and imposing the condition that every edge lies either in the interior or in the exterior of the circle (see Figure 1). In this paper we shall often use this equivalent *circular model* for 2-page drawings, as well as the usual *spine model*. It is known that the family of graphs G with $\nu_2(G) = 0$ is precisely the family

*Received by the editors October 18, 2011; accepted for publication (in revised form) March 27, 2012; published electronically June 5, 2012.
http://www.siam.org/journals/siopt/22-2/85220.html

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of subgraphs of Hamiltonian planar graphs [2]. As a consequence, there exist planar graphs G with $\nu_2(G) > 0$, in contrast to the case of the normal crossing number. In fact, it was shown that all planar graphs may be embedded without crossings in 4-page books, and that four pages are necessary [35].

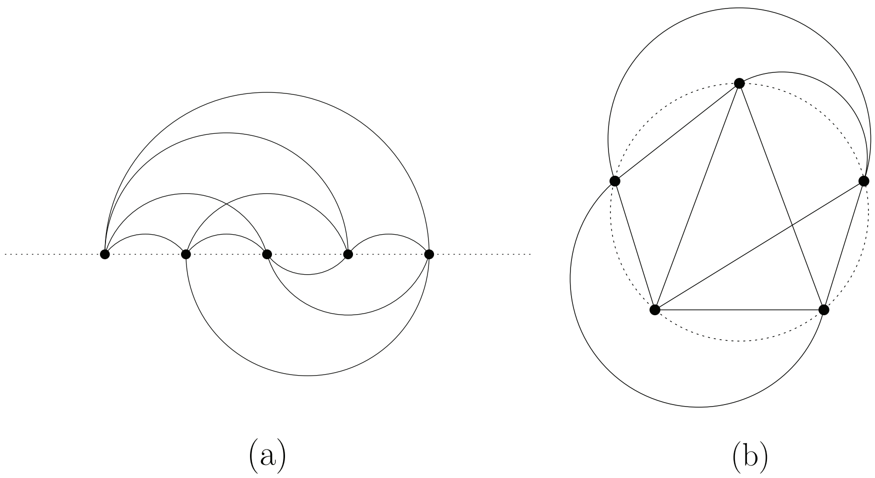

FIG. 1. *A 2-page drawing of K_5:* (a) *in the spine model; and* (b) *in the circular model.*

Masuda et al. [25, 26] proved that the decision problems for ν_1 and ν_2 are NP-complete. Shahrokhi et al. [30] gave an approximation algorithm for $\nu_k(G)$, as well as applications to the rectilinear crossing number. A more recent, additional motivation for studying k-page crossing numbers comes from Quantum Dot Cellular Automata [32].

Several interesting algorithms and heuristics have been proposed for producing 1- and 2-page drawings (see, for instance, [5, 6, 16, 17, 18, 19]). As with the usual crossing number, the exact computation of $\nu_k(G)$ (for any integer k) is a very challenging problem, even for restricted families of graphs. In this direction, Fulek et al. [7], He, Sălăgean, and Mäkinen [15], and Riskin [29] have computed the exact 1-page and 2-page crossing numbers of several interesting families of graphs.

1.1. Drawings of $K_{m,n}$ and K_n. Turán asked in the 1940's, What is the crossing number of the complete bipartite graph $K_{m,n}$? There is a natural drawing of $K_{m,n}$ with exactly $Z(m,n) := \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor$ crossings (see Figure 2), and so $\mathrm{cr}(K_{m,n}) \le Z(m,n)$.

Perhaps the foremost open crossing number problem is *Zarankiewicz's Conjecture*, dating back to the early 1950's [36]:

$$\mathrm{cr}(K_{m,n}) \stackrel{?}{=} Z(m,n). \tag{1}$$

This conjecture has been verified only for $\min\{m,n\} \le 6$ [20], and for the special cases $(m,n) \in \{(7,7),(7,8),(7,9),(7,10),(8,8),(8,9),(8,10)\}$ [34].

On a parallel front, there are drawings of the complete graph K_n with exactly $Z(n) := \frac{1}{4} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \lfloor \frac{n-2}{2} \rfloor \lfloor \frac{n-3}{2} \rfloor$ crossings (for every n), and so $\mathrm{cr}(K_n) \le Z(n)$. These drawings inspired the still open, long-standing Harary–Hill conjecture [13]:

$$\mathrm{cr}(K_n) \stackrel{?}{=} Z(n). \tag{2}$$

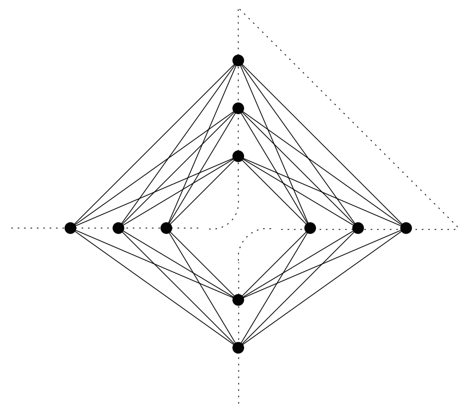

FIG. 2. *A drawing of* $K_{5,6}$ *with* $Z(5,6) = 24$ *crossings. By performing a homeomorphism from the plane to itself that takes the dotted curve to a straight line, the result is a* 2*-page drawing of* $K_{5,6}$ *with the same number of crossings.*

This conjecture has been verified for $n \leq 12$ [27].

For a detailed account on the history of (1) and (2), we refer the reader to the lively survey by Beineke and Wilson [1].

1.2. 2-page drawings of $K_{m,n}$ and K_n. The drawing in Figure 2 is easily generalized to yield a drawing of $K_{m,n}$ with $Z(m,n)$ crossings. As mentioned in the caption of this figure, such a drawing is easily transformed into a 2-page drawing of $K_{m,n}$ with the same number of crossings. Thus, there exist 2-page drawings of $K_{m,n}$ with $Z(m,n)$ crossings.

On the other hand, it is somewhat surprising that there exist 2-page drawings of K_n with exactly $Z(n)$ crossings, for every positive integer n (see [12]; see also [14]).

These observations imply that $\nu_2(K_{m,n}) \leq Z(m,n)$ and $\nu_2(K_n) \leq Z(n)$. Since obviously $\mathrm{cr}(G) \leq \nu_2(G)$ for every graph G, (1) and (2) immediately imply the following conjectures:

$$\nu_2(K_{m,n}) \stackrel{?}{=} Z(m,n), \tag{3}$$

$$\nu_2(K_n) \stackrel{?}{=} Z(n). \tag{4}$$

Even though (3) and (4) are (at least in principle) weaker than the corresponding (1) and (2), and even though the 2-page crossing number problem can be naturally formulated in purely combinatorial terms, our current knowledge (prior to this paper) on (3) and (4) is not substantially better than our knowledge on (1) and (2). Indeed, the only step ahead is the proof by Buchheim and Zheng [3] that $\nu_2(K_{13}) = Z(13)$ (from which a routine counting argument yields that $\nu_2(K_{14}) = Z(14)$). The best general lower bounds known for $\nu_2(K_{m,n})$ and $\nu_2(K_n)$ are the same as those known for $\mathrm{cr}(K_{m,n})$ and $\mathrm{cr}(K_n)$, and the same is true for the asymptotic ratio $\lim_{n\to\infty} \nu_2(K_n)/Z(n)$, whose best current estimate is exactly the same as the asymptotic ratio $\lim_{n\to\infty} \mathrm{cr}(K_n)/Z(n)$, namely 0.859 [21].

1.3. Main results. Our main results in this paper offer a substantial improvement on our knowledge of $\nu_2(K_{m,n})$ and $\nu_2(K_n)$ over our knowledge of $\mathrm{cr}(K_{m,n})$ and $\mathrm{cr}(K_n)$.

THEOREM 1.1. *The* 2*-page Harary–Hill conjecture holds for all* $m \leq 18$ *and for* $m = 20$ *and* 24*:*

$$\nu_2(K_m) = Z(m)\ \forall\ m \leq 18 \quad \text{and for} \quad m \in \{20, 24\}. \tag{A}$$

Moreover, the asymptotic ratio between the 2*-page crossing number of* K_n *and its conjectured value satisfies*

$$\lim_{n\to\infty} \frac{\nu_2(K_n)}{Z(n)} \geq 0.9253. \tag{B}$$

THEOREM 1.2. *The* 2*-page Zarankiewicz's conjecture holds in the asymptotically relevant term for* $m = 7$ *and* 8*. That is*

$$\nu_2(K_{7,n}) = (9/4)n^2 + O(n) = Z(7,n) + O(n)$$

and

$$\nu_2(K_{8,n}) = 3n^2 + O(n) = Z(8,n) + O(n).$$

Therefore,

$$\lim_{n\to\infty} \frac{\mathrm{cr}(K_{7,n})}{Z(7,n)} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\mathrm{cr}(K_{8,n})}{Z(8,n)} = 1.$$

Outline of this paper. The rest of this paper is structured as follows. In section 2, we review the reformulation (first unveiled by Buchheim and Zheng [3]) in which the problem of calculating $\nu_2(K_n)$ is shown to be equivalent to a maximum cut problem on an associated graph G_n. In section 3 we invoke a result by Goemans and Williamson that provides an upper bound on the size of the maximum cut of a graph; this bound may be computed via semidefinite programming. Using these ingredients, in section 4 we present the numerical computations that establish Theorem 1.1. In section 5 we formulate a quadratic program whose solution yields a lower bound on $\nu_2(K_{m,n})$. In section 6 we analyze the semidefinite programming relaxation of this quadratic program, and in section 7 we give the numerical computations that prove Theorem 1.2. In section 8 we present some concluding remarks.

2. Formulating $\nu_2(K_n)$ as a maximum cut problem. Buchheim and Zheng [3] unveiled a natural reformulation of the fixed linear crossing number problem (FLCNP) as a maximum cut problem. Their results imply, in particular, that $\nu_2(K_n)$ can be obtained by computing the maximum cut size in a certain graph $G_n = (V_n, E_n)$, with V_n and E_n defined as follows.

Consider a Hamiltonian cycle with vertices $v_1, v_2, \ldots, v_n$. Let V_n be the set of *chords* of the cycle; that is, the edges v_iv_j with v_i and v_j at cyclic distance at least 2. Now to define E_n, let two chords v_iv_j and v_kv_ℓ be adjacent if they intersect. This construction is illustrated in Figure 3 for $n = 5$.

Thus $|V_n| = \binom{n}{2} - n$, and it is easy to check that $|E_n| = \binom{n}{4}$. The automorphism group of G_n is isomorphic to the dihedral group D_n, and there are $d - 1$ orbits of vertices, where $d = \lfloor n/2 \rfloor$. The equivalency classes of vertices (i.e., vertices belonging to the same orbit) may be described as follows: since vertices correspond to chords in P_n, the chords that connect vertices of P_n at the same cyclic distance belong to the

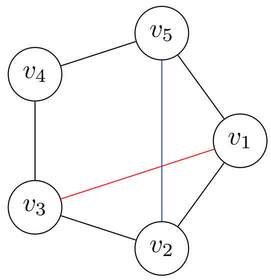

FIG. 3. *The chords v_1v_3 and v_2v_5 form adjacent vertices in the graph G_5.*

same equivalency class. The vertices corresponding to chords with cyclic distance i have valency $i(i-1)+2(i-1)(d-i)$, as is easy to check.

Now for a graph $G=(V,E)$ and a subset $W \subset V$, $\mathrm{cut}_W(G)$ denotes the number of edges with precisely one endpoint in W, and $\mathrm{maxcut}(G)$ is the maximum value of $\mathrm{cut}_W(G)$ taken over all subsets $W \subset V$.

The next lemma follows immediately from Theorem 1 in [3]. We sketch the proof for the sake of completeness.

LEMMA 2.1.

$$\nu_2(K_n) = |E_n| - \mathrm{maxcut}(G_n).$$

Proof. Given a 2-page (circle) drawing of K_n, define $W \subset V_n$ as the chords that are drawn inside the circle. The edges of E_n with precisely one endpoint in W now correspond to edges of K_n that do not cross in the drawing. □

As a consequence of this lemma, one may calculate $\nu_2(K_n)$ for fixed (in practice, sufficiently small) values of n by solving a maximum cut problem. This was done by Buchheim and Zheng [3] for $n \le 13$, by solving the maximum cut problem with a branch-and-bound algorithm (Bucheim and Zheng applied the technique to many other graphs as well). Using the `BiqMac` solver [28], we have computed the exact value of $\nu_2(K_n)$ for $n \le 18$ and for $n \in \{20, 24\}$ (statement (A) in Theorem 1.1; see section 4).

3. The Goemans–Williamson max-cut bound. We follow the standard practice to use $\mathbb{R}^{p\times q}$ (respectively, $\mathbb{C}^{p\times q}$) to denote the space of $p \times q$ matrices over $\mathbb{R}$ (respectively, $\mathbb{C}$). For $\mathbf{A} \in \mathbb{R}^{p\times p}$, the notation $\mathbf{A} \succeq 0$ means that $\mathbf{A}$ is symmetric positive semidefinite, whereas for $\mathbf{A} \in \mathbb{C}^{p\times p}$, it means that $\mathbf{A}$ is Hermitian positive semidefinite.

Let G be a graph with p vertices, and let $\mathbf{L}$ be its Laplacian matrix. Goemans and Williamson [9] introduced the following semidefinite programming-based upper bound on $\mathrm{maxcut}(G)$:

$$\mathrm{maxcut}(G) \le \mathcal{GW}(G) := \max\left\{ \frac{1}{4}\mathrm{trace}(\mathbf{LX}) \;\middle|\; \mathbf{X} \succeq 0,\ X_{ii} = 1\ (1 \le i \le p) \right\}. \tag{5}$$

It was shown in [9] that $0.878\mathcal{GW}(G) \le \mathrm{maxcut}(G) \le \mathcal{GW}(G)$ holds for all graphs G.

The associated dual semidefinite program takes the form

$$\mathcal{GW}(G) = \min_{\mathbf{w}\in\mathbb{R}^p} \left\{ \sum_i w_i \;\middle|\; \mathrm{Diag}(\mathbf{w}) - \frac{1}{4}\mathbf{L} \succeq 0 \right\}, \tag{6}$$

where Diag is the operator that maps a p-vector to a $p \times p$ diagonal matrix in the obvious way.

3.1. The Goemans–Williamson bound for G_n. Using the technique of symmetry reduction for semidefinite programming (see, e.g., [8]), one can simplify the dual problem (6) for the graphs G_n defined in section 2, by using the dihedral automorphism group of G_n. We state the final expression as the following lemma.

LEMMA 3.1. *Let $n > 0$ be an odd integer and let $d = \lfloor n/2 \rfloor$. One has*

$$\mathcal{GW}(G_n) = \min_{y \in \mathbb{R}^{d-1}} \left\{ n \sum_{i=2}^{d} y_i \;\middle|\; \mathrm{Diag}\left(y - \frac{1}{4} val\right) + \Lambda^{(m)} \succeq 0 \ (0 \leq m \leq d) \right\},$$

where

$$\begin{aligned} &val_i = i(i-1) + 2(i-1)(d-i), \quad 2 \leq i \leq d,\\ &\Lambda^{(m)}_{ij} = \frac{1}{4}\left[\sum_{k=1}^{i-1} e^{\frac{-2\pi m k\sqrt{-1}}{n}} + \sum_{k=n-j+1}^{n-j+i-1} e^{\frac{-2\pi m k\sqrt{-1}}{n}}\right], \quad 2 \leq i \leq j \leq d,\\ &\Lambda^{(m)} = \Lambda^{(m)^*} \in \mathbb{C}^{d-1 \times d-1}. \end{aligned} \tag{7}$$

For the proof, we recall that the *Kronecker product* $\mathbf{A} \otimes \mathbf{B}$ of matrices $\mathbf{A} \in \mathbb{R}^{p \times q}$ and $\mathbf{B} \in \mathbb{R}^{r \times s}$ is defined as the $pr \times qs$ matrix composed of pq blocks of size $r \times s$, with block ij given by $a_{ij}\mathbf{B}$ where $1 \leq i \leq p$ and $1 \leq j \leq q$.

Proof. We first label the vertices G_n as follows. Consider the cycle C_n with vertices numbered $\{0, 1, \ldots, n-1\}$ in the usual way. The vertices of G_n that correspond to chords connecting points at cyclic distance i are now given consecutive labels $(0, i), (1, i+1), \ldots (n-1, i-1)$. Thus the adjacency matrix of G_n is partitioned into a block structure, where each row of blocks is indexed by a cyclic distance $i \in \{2, \ldots, d\}$, and each block has size $n \times n$.

Moreover, block (i, j) $(i, j \in \{2, \ldots, d\},\ i \leq j)$ is given by the $n \times n$ circulant matrix with first row

$$[0\ \mathbf{1}^T_{i-1}\ \mathbf{0}^T_{n-i-j+1}\ \mathbf{1}^T_{i-1}\ \mathbf{0}^T_{j-i}],$$

where $\mathbf{1}_k$ and $\mathbf{0}_k$ denote the all-ones and all-zeroes vectors in $\mathbb{R}^k$, respectively.

The eigenvalues of this block are

$$\lambda_m = \sum_{k=1}^{i-1} e^{\frac{-2\pi m k\sqrt{-1}}{n}} + \sum_{k=n-j+1}^{n-j+i-1} e^{\frac{-2\pi m k\sqrt{-1}}{n}} \qquad (0 \leq m \leq n-1); \tag{8}$$

see, e.g., [11].

Now let an optimal solution $\mathbf{w}$ of the semidefinite program (6) be given for $G = G_n$. If we project the matrix

$$\mathrm{Diag}(\mathbf{w}) + \frac{1}{4}\mathbf{L}$$

onto the centralizer ring of $\mathrm{Aut}(G_n)$, then we again obtain an optimal solution. Indeed, this projection simply averages the components of w over the $d-1$ orbits of $\mathrm{Aut}(G_n)$. Moreover, the projection is also a symmetric positive semidefinite matrix, since any projection of a Hermitian positive semidefinite matrix onto a matrix $*$-algebra is again positive semidefinite (see, e.g., [10]).

Denoting the average of the w components in orbit i by y_i, we obtain an optimal solution of the form

$$\mathcal{GW}(G_n) = \min_{\mathbf{y}\in\mathbb{R}^{d-1}} n \sum_{i=2}^{d} y_i$$

such that

$$\sum_{i=2}^{d} y_i \left(\mathbf{e}_{i-1}\mathbf{e}_{i-1}^T\right) \otimes \mathbf{I}_n - \frac{1}{4}\mathbf{L} \succeq 0, \tag{9}$$

where $\mathbf{e}_i$ denotes the ith standard unit vector in $\mathbb{R}^{d-1}$, and $\mathbf{I}_n$ denotes the identity matrix of order n.

Let $\mathbf{Q}$ denote the (unitary) discrete Fourier transform matrix of order n. Condition (9) is equivalent to

$$(\mathbf{I}_n \otimes \mathbf{Q}) \left(\sum_{i=2}^{d} y_i \left(\mathbf{e}_{i-1}\mathbf{e}_{i-1}^T\right) \otimes \mathbf{I}_n - \frac{1}{4}\mathbf{L}\right) (\mathbf{I}_n \otimes \mathbf{Q})^* \succeq 0. \tag{10}$$

Since the unitary transform involving $\mathbf{Q}$ diagonalizes any circulant matrix (see, e.g., [11]), the matrix $(\mathbf{I}_n \otimes \mathbf{Q})\mathbf{L}(\mathbf{I}_n \times \mathbf{Q})^*$ becomes a block matrix where each $n \times n$ block is diagonal, with diagonal entries of block (i, j) given by the eigenvalues in (8).

Finally, the rows and columns of the left-hand side of (10) may now be reordered to form a block diagonal matrix with $n \times n$ diagonal blocks given by the right-hand side of (7) (only $d + 1$ of these blocks are distinct). This completes the proof. ∎

A few remarks on the semidefinite programming reformulation in Lemma 3.1:

- The constraints involve Hermitian (complex) linear matrix inequalities, as opposed to the real symmetric linear matrix inequalities in (6).
- The reduced problem has $d + 1$ linear matrix inequalities involving $(d - 1) \times (d - 1)$ matrices. By comparison, the original problem had one linear matrix inequality involving $(\binom{n}{2} - n) \times (\binom{n}{2} - n)$ matrices. As a result, the reformulation of $\mathcal{GW}(G_n)$ may be solved for much larger values of n than the original formulation (6) (see next section).
- Although we have only done the symmetry reduction of problem (6) for G_n with n odd, the case for even n is similar, but omitted, since we will not use it later.
- Any feasible point $y \in \mathbb{R}^{d-1}$ of the reduced problem in Lemma 3.1 provides a certificate of an upper bound on $\mathcal{GW}(G_n)$, and consequently a certificate of a lower bound on $\nu_2(K_n)$, since $\nu_2(K_n) \geq \binom{n}{4} - \mathcal{GW}(G_n)$.

4. Numerical computations: Proof of Theorem 1.1. Theorem 1.1 (A) follows by an exact computation of the related maxcut problem of G_n for certain values of n, while Theorem 1.1 (B) follows by a calculation of $\mathcal{GW}(G_{899})$ and a standard counting argument.

4.1. Proof of (A). First we observe that if $n < 5$, then $Z(n) = 0$, and the assertion $\nu_2(K_n) = Z(n)$ is easily verified.

We computed the exact value $\mathrm{maxcut}(G_n)$ for $n = 5, 7, 9, 11, 13, 15, 17, 20$, and 24, using the solver `BiqMac` [28], available from http://biqmac.uni-klu.ac.at/. Computation was done on a quad-core 2.0 GHz Intel PC with 10 GB of RAM memory, running Linux. We used a cut-off time of 60 hours for the computation for each

TABLE 1

The second column gives the exact values of maxcut(G_n) that we computed. The fourth column gives the corresponding exact values of $\nu_2(K_n)$ (using that $\nu_2(K_n) = |E_n| - \text{maxcut}(G_n)$). For all these values of n, the conjecture $\nu_2(K_n) = Z(n)$ is verified.

n	maxcut(G_n)	$\lvert E_n\rvert = \binom{n}{4}$	$\nu_2(K_n)$	$Z(n)$	CPU time(s)	Branch and bound nodes
5	4	5	1	1	0.001	1
7	26	35	9	9	0.01	1
9	90	126	36	36	0.22	3
11	230	330	100	100	4.01	17
13	490	715	225	225	73.27	151
15	924	1,365	441	441	906.61	841
17	1,596	2,380	784	784	15,542	6,837
20	3,225	4,845	1,620	1,620	58,784	9,479
24	6,996	10,626	3,630	3,630	5,616	65

value of n. As a consequence, the `BiqMac` solver failed to terminate successfully in a few cases, namely $n = 19, 21, 22$, and 23. In general, we observed that the `BiqMac` solver performed better for even values of n that for odd values. We do not have an explanation for this behavior.

The results are presented in the second column of Table 1. The exact value of $\nu_2(K_n)$ (fourth column) follows from the second and third columns (using Lemma 2.1). The fifth column is given for reference, to verify that $\nu_2(K_n) = Z(n)$ for all these values of n. Thus (A) follows for $n = 5, 7, 9, 11, 13, 15, 17, 20$, and 24. The last two columns show the CPU time required, and the number of nodes evaluated in the branch and bound tree by the solver `BiqMac`. Note that the computation succeeded for $n = 24$ (as opposed to $n = 19, 21, 22$, and 23), since only 65 branching nodes were needed in this case. Finally, an elementary, well-known counting argument shows that if $\nu_2(K_{2m+1}) = Z(2m + 1)$ for some positive integer m, then $\nu_2(K_{2m+2}) = Z(2m + 2)$. This proves (A) for the remaining cases $n = 6, 8, 10, 12, 14, 16$, and 18.

4.2. Proof of (B). The first ingredient in the proof of (B) is a lower bound for $\nu_2(K_{899})$. We obtained this bound via the approximate calculation of $\mathcal{GW}(G_{899})$, which we achieved by using the semidefinite programming reformulation in Lemma 3.1. Computation was done on a Dell Precision T7500 workstation with 92GB of RAM, using the semidefinite programming solver SDPT3 [31, 33] under MATLAB 7 together with the MATLAB package YALMIP [24]. The total running time was 12,602 seconds. SDPT3 was chosen since it can deal with Hermitian matrix variables. We obtained $\mathcal{GW}(G_{899}) \le 1.76537474 \times 10^{10}$. Using Lemma 2.1 and (5), it follows immediately that

$$\nu_2(K_{899}) \ge 9{,}381{,}181{,}976. \tag{11}$$

The second ingredient to prove (B) is to establish a lower bound on the asymptotic ratio $\lim_{n\to\infty} \nu_2(K_n)/Z(n)$ that can be guaranteed from a lower bound on $\nu_2(K_m)$ for some $m > 3$.

CLAIM 4.1. *For any integer $m > 3$,*

$$\lim_{n\to\infty} \frac{\nu_2(K_n)}{Z(n)} \ge \frac{64}{m(m-1)(m-2)(m-3)}\,\nu_2(K_m).$$

Proof. Let m, n be integers with $3 < m < n$. Consider a 2-page drawing D of K_n with $\nu_2(K_n)$ edge crossings. Let $\mathcal{G}$ denote the set of subgraphs of K_n that are isomorphic to K_m; i.e., $|\mathcal{G}| = \binom{n}{m}$. Any two disjoint edges in K_n occur in $\binom{n-4}{m-4}$ of the graphs in $\mathcal{G}$. Thus, every crossing in D appears in the induced drawings of $\binom{n-4}{m-4}$ graphs in $\mathcal{G}$. Consequently,

$$\nu_2(K_n) \geq \frac{\nu_2(K_m)\binom{n}{m}}{\binom{n-4}{m-4}} = \frac{\nu_2(K_m)n(n-1)(n-2)(n-3)}{m(m-1)(m-2)(m-3)}.$$

The claim follows immediately from this inequality and the definition of $Z(n)$. ∎

It only remains to observe that (B) is an immediate consequence of (11) and Claim 4.1.

5. A quadratic programming lower bound for $\boldsymbol{\nu_2(K_{m,n})}$. Throughout this section, assume that m is fixed, and consider 2-page drawings of $K_{m,n}$, where n is any positive integer. Thus, all vertices lie on the x-axis, and each edge is contained either in the upper or in the lower half-plane. We assume, without any loss of generality, that the m degree-n *blue* vertices $b_1, b_2, \ldots, b_m$ appear on the x-axis in this order, from left to right. The n degree-m vertices are *red*. The *star* of a red vertex r (which we shall denote $\mathrm{star}(r)$) is the subgraph induced by r and its incident edges. Thus, for every red vertex r, $\mathrm{star}(r)$ is isomorphic to $K_{m,1}$.

5.1. The type of a red vertex. In our quest for lower bounding the number of crossings in any 2-page drawing $\mathcal{D}$ of $K_{m,n}$, the strategy is to consider any two red vertices r, r', and find a lower bound for the number $\times_{\mathcal{D}}(\mathrm{star}(r), \mathrm{star}(r'))$ of crossings in $\mathcal{D}$ that involve one edge in $\mathrm{star}(r)$ and one edge in $\mathrm{star}(r')$. The bound we establish is in terms of the *types* of r and r'. The type (formally defined shortly) of a red vertex is determined by its position relative to the blue vertices, and by which edges incident with it lie on each half-plane.

We start by noting that we may focus our interest in drawings in which no red vertex lies to the left of b_1. Indeed, if the leftmost red vertex lies to the left of b_1 (and so it is the leftmost vertex overall), it is easy to see that it may be moved so that it becomes the rightmost (overall) vertex, without increasing the number of crossings. By repeating this procedure we get a drawing with the same number of crossings, and with no red vertex to the left of b_1. Thus there is no loss of generality in dealing only with drawings that satisfy this property, and it follows that each red vertex r has a *position* $p(r)$ relative to the blue points: $p(r)$ is the largest $j \in \{1, 2, \ldots, m\}$ such that r is to the right of b_j.

Also, to each red vertex r we can naturally assign a partition $\{U(r), L(r)\}$ of $\{1, 2, \ldots, m\}$, the *distribution* of r, defined by the rule that $j \in \{1, 2, \ldots, m\}$ is in $U(r)$ (respectively, $L(r)$) if the edge rb_j lies in the upper (respectively, lower) half-plane. We call the triple $(p(r), U(r), L(r))$ the *type* of r, and denote it by $\mathrm{type}(r)$. Since $p(r)$ can be any integer in $\{1, 2, \ldots, m\}$, and $U(r)$ any subset of $\{1, 2, \ldots, m\}$ (and $L(r) = \{1, 2, \ldots, m\} \setminus U(r)$ is determined by $U(r)$), it follows that there are $m2^m$ possible types for a red vertex. We use $\mathrm{Types}(m)$ to denote the collection of all $m2^m$ possible types.

5.2. Guaranteeing crossings between red stars using types. The motivation for introducing the concept of type is that knowing the types of two red vertices r and r' in a drawing $\mathcal{D}$ of $K_{m,n}$ yields a lower bound on $\times_{\mathcal{D}}(\mathrm{star}(r), \mathrm{star}(r'))$.

We illustrate this with an example. Suppose that $m = 5$, and that $\mathrm{type}(r) = (2, \{1, 2, 3, 5\}, \{4\})$ and $\mathrm{type}(r') = (3, \{1, 3, 4, 5\}, \{2\})$. The situation is thus as illustrated in Figure 4.

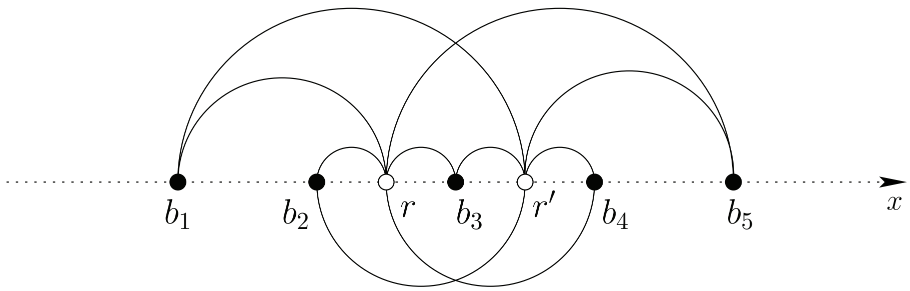

FIG. 4. *The types of red vertices r and r' are* $(2, \{1, 2, 3, 5\}, \{4\})$ *and* $\mathrm{type}(r') = (3, \{1, 3, 4, 5\}, \{2\})$*, respectively. Thus, r is in position 2 (that is, between b_2 and b_3), and the edges joining r to b_1, b_2, b_3, and b_5 are in the upper half-plane and the edge joining r to b_4 is in the lower half-plane. Both crossings in this drawing can be easily predicted from* $\mathrm{type}(r)$ *and* $\mathrm{type}(r')$.

Both crossings between $\mathrm{star}(r)$ and $\mathrm{star}(r')$ in this example are easily detected from $\mathrm{type}(r)$ and $\mathrm{type}(r')$. Indeed, since b_1, r, r', b_5 occur in this order from left to right (this follows since r and r' are in positions 2 and 3, respectively), and b_1r' and rb_5 are both on the upper half-plane (this follows since $1 \in U(r')$ and $5 \in U(r)$), it follows that b_1r' and rb_5 must cross. We remark that the key pieces of information are that (i) the endpoints b_1, r, r', b_5 of b_1r' and rb_5 *alternate* on the x-axis (that is, they are all distinct and occur in the x-axis so that the ends of one edge are in first and third place and the ends of the other edge are in second and fourth place); and (ii) both edges are drawn on the same half-plane.

Using this simple criterion (if two edges are on the same half-plane and their endpoints alternate, then they must cross each other), given two red points r, r' in a drawing $\mathcal{D}$ of $K_{m,n}$, it is easy to derive a lower bound for $\times_{\mathcal{D}}(\mathrm{star}(r), \mathrm{star}(r'))$ in terms of $\mathrm{type}(r)$ and $\mathrm{type}(r')$. This bound (Proposition 5.1) is given in terms of a quantity we now proceed to define.

First, for $\sigma = (p, U, L)$ and $\tau = (p', U', L') \in \mathrm{Types}(m)$, we let

$$[\sigma, \tau] := \Bigg| \Bigg\{ (i, j) \;\Bigg|\; \Big(\big(i \in U \text{ and } j \in U'\big) \text{ or } \big(i \in L \text{ and } j \in L'\big) \Big) \text{ and } \\ \Big(\big(i < j \le p\big) \text{or} \big(j \le p \text{and} p' < i\big) \text{or } \big(i < j \text{ and} p' < i\big) \text{ or} \big(p < j < i \le p'\big) \Big) \Bigg\} \Bigg|,$$

and

$$Q_{\sigma\tau} := \begin{cases} [\sigma, \tau] & \text{if } p < p', \\ [\tau, \sigma] & \text{if } p > p', \\ \min\big\{[\sigma, \tau], [\tau, \sigma]\big\} & \text{if } p = p'. \end{cases}$$

The nonnegative integers $Q_{\sigma\tau}$ can be naturally regarded as the entries of a $m2^m \times m2^m$-matrix $\mathbf{Q}$ indexed (both by rows and columns) by the elements of $\mathrm{Types}(m)$. It is easy to check that the matrix $\mathbf{Q}$ is symmetric, and its entries provide the lower bounds we have been aiming for.

PROPOSITION 5.1. *Let $\sigma, \tau \in \mathrm{Types}(m)$, and suppose that r_σ, r_τ are red points in a drawing $\mathcal{D}$ of $K_{m,n}$, such that* $\mathrm{type}(r_\sigma) = \sigma$ *and* $\mathrm{type}(r_\tau) = \tau$. *Then*

$$\times_{\mathcal{D}}(\mathrm{star}(r_\sigma), \mathrm{star}(r_\tau)) \geq Q_{\sigma\tau}.$$

Proof. Suppose first that r_σ occurs to the left of r_τ. It is easy to verify that if i, j are integers such that either (i) $i < j \leq p$; (ii) $j \leq p$ and $p' < i$; or (iii) $i < j$ and $p' < i$; or (iv) $p < j < i \leq p'$, then the endpoints of rb_i and $r'b_j$ alternate. Therefore, if either $i \in U$ and $j \in U'$, or $i \in L$ and $j \in L'$, then rb_i and $r'b_j$ cross each other. Therefore there is an injection from the set of all pairs (i, j) of integers that satisfy the condition in the definition of $[\sigma, \tau]$, to the set of crossings that involve an edge in $\mathrm{star}(r_\sigma)$ and an edge in $\mathrm{star}(r_\tau)$; that is, $\times_{\mathcal{D}}(\mathrm{star}(r_\sigma), \mathrm{star}(r_\tau)) \geq [\sigma, \tau]$.

Similarly, if r_σ occurs to the right of r_τ, then $\times_{\mathcal{D}}(\mathrm{star}(r_\sigma), \mathrm{star}(r_\tau)) \geq [\tau, \sigma]$.

Now if $p < p'$ (respectively, $p > p'$), then r_σ necessarily occurs to the left (respectively, to the right) of r_τ, and so it follows that $\times_{\mathcal{D}}(\mathrm{star}(r_\sigma), \mathrm{star}(r_\tau)) \geq [\sigma, \tau] = Q_{\sigma\tau}$ (respectively, $\geq [\tau, \sigma] = Q_{\sigma\tau}$), as required. Finally, if $p = p'$, then r_σ can be either to the right or to the left of r_τ. In the first case, $\times_{\mathcal{D}}(\mathrm{star}(r_\sigma), \mathrm{star}(r_\tau)) \geq [\sigma, \tau]$, while in the second case $\times_{\mathcal{D}}(\mathrm{star}(r_\sigma), \mathrm{star}(r_\tau)) \geq [\tau, \sigma]$. Thus, in this case, $\times_{\mathcal{D}}(\mathrm{star}(r_\sigma), \mathrm{star}(r_\tau)) \geq \min\{[\sigma, \tau], [\tau, \sigma]\} = Q_{\sigma\tau}$, as required. ∎

5.3. The quadratic program. Consider now any fixed 2-page drawing $\mathcal{D}$ of $K_{m,n}$. For each type $\sigma \in \mathrm{Types}(m)$, let n_σ denote the number of red vertices whose type in $\mathcal{D}$ is σ, let $p_\sigma := n_\sigma/n$, and let $\mathbf{p}$ be the vector $(p_\sigma)_{\sigma \in \mathrm{Types}(m)}$. It follows immediately from Proposition 5.1 that the number $\nu_2(\mathcal{D})$ of crossings in $\mathcal{D}$ satisfies

$$\begin{aligned}
\nu_2(\mathcal{D}) &\geq \frac{1}{2} \sum_{\substack{\sigma,\tau \in \mathrm{Types}(m) \\ \sigma \neq \tau}} Q_{\sigma\tau} n_\sigma n_\tau + \sum_{\sigma \in \mathrm{Types}(m)} Q_{\sigma\sigma} \binom{n_\sigma}{2} \\
&= \frac{1}{2} \sum_{\sigma,\tau \in \mathrm{Types}(m)} Q_{\sigma\tau} n_\sigma n_\tau - \frac{1}{2} \sum_{\sigma \in \mathrm{Types}(m)} Q_{\sigma\sigma} n_\sigma \\
&= \frac{n^2}{2} \mathbf{p}^T \mathbf{Q} \mathbf{p} - \frac{n}{2} \sum_{\sigma \in \mathrm{Types}(m)} Q_{\sigma\sigma} p_\sigma \\
&\geq \frac{n^2}{2} \mathbf{p}^T \mathbf{Q} \mathbf{p} - \frac{n}{2} \max_{\sigma \in \mathrm{Types}(m)} Q_{\sigma\sigma} \\
&\geq \frac{n^2}{2} \mathbf{p}^T \mathbf{Q} \mathbf{p} - \frac{m(m-1)n}{4},
\end{aligned}$$

where for the last inequality we use that $\Sigma_{\sigma \in \mathrm{Types}(m)} p_\sigma = 1$ and that $Q_{\sigma\sigma} = [\sigma, \sigma] \leq \binom{m}{2}$.

The derived inequality holds for every 2-page drawing $\mathcal{D}$ of $K_{m,n}$, and so in particular for a crossing-minimal drawing. Thus, if we let

$$\Delta = \left\{ \mathbf{x} = (x_1, x_2, \ldots, x_{m2^m})^T \in \mathbb{R}^{m2^m} \;\middle|\; \sum_i x_i = 1,\ x_i \geq 0 \right\}$$

denote the standard simplex in $\mathbb{R}^{m2^m}$, then we obtain

$$\nu_2(K_{m,n}) \geq \frac{n^2}{2} \left(\min_{\mathbf{x} \in \Delta} \mathbf{x}^T \mathbf{Q} \mathbf{x} \right) - \frac{m(m-1)n}{4}. \tag{12}$$

We may therefore obtain a lower bound on $\nu_2(K_{m,n})$ for some fixed m (we will be particularly interested in the case $m = 7$), by solving the standard quadratic programming problem

$$\mathrm{lb}(m) = \min_{\mathbf{x}\in\Delta} \mathbf{x}^T\mathbf{Q}\mathbf{x}. \tag{13}$$

The standard quadratic programming problem is NP-hard in general, and we will compute only a lower bound on the minimum via semidefinite programming, as explained in the next section.

6. A semidefinite programming lower bound on $\nu_2(K_{m,n})$. The usual semidefinite programming relaxation of problem (13) takes the form

$$\begin{aligned} \mathrm{lb}(m) &\geq \min\left\{\mathrm{trace}(\mathbf{QX}) \;\middle|\; \mathrm{trace}(\mathbf{JX}) = 1,\ \mathbf{X} \succeq \mathbf{0},\ \mathbf{X} \geq \mathbf{0}\right\} \\ &= \max\left\{t \;\middle|\; \mathbf{Q} - t\mathbf{J} = \mathbf{S_1} + \mathbf{S_2},\ \mathbf{S_1} \succeq \mathbf{0},\ \mathbf{S_2} \geq \mathbf{0}\right\} \\ &:= \mathrm{SDP}_{bound}(m), \end{aligned} \tag{14}$$

where $\mathbf{J}$ is the all-ones matrix, and $\mathbf{X} \geq \mathbf{0}$ means that $\mathbf{X}$ is entrywise nonnegative. We observe that the first equality is due to the duality theory of semidefinite programming.

Due to the special structure of $\mathbf{Q}$, we may again use symmetry reduction to reduce the size of these problems. To this end, for odd m, we may order the rows and columns of $\mathbf{Q}$ to obtain a block matrix consisting of circulant blocks of order $2m$. (Thus there are 2^{m-1} rows/columns of blocks.) The ordering of rows works as follows: we first define a group action on the set Types(m). For ease of notation we now represent the elements of Types(m) as (p, U), with $p \in \{0, \ldots, m-1\}$ and $U \subseteq \{0, \ldots, m-1\}$, i.e., we now number the m vertices from 0 to $m-1$, and omit the set L (which is redundant in the description since it is the complement of U).

The group in question is generated by the following two elements, a "flip":

$$g_1 : (p, U) \mapsto (p, \{0, \ldots, m-1\} \setminus U),$$

and a "cyclic shift":

$$g_2 : (p, U) \mapsto (p + 1 \mod m, \{u + 1 \mod m \mid u \in U\}).$$

Note that g_1 and g_2 commute and therefore generate an Abelian group of order $2m$. If m is odd, then $g := g_1 \circ g_2$ generates the entire group, i.e., in this case we obtain the cyclic group of order $2m$. Indeed, the order of g equals the least common multiple of the orders of g_1 and g_2, namely $2m$ if m is odd.

Also note that

$$Q_{\sigma,\tau} = Q_{g_i(\sigma),g_i(\tau)} \quad \forall \sigma, \tau \in \mathrm{Types}(m),\ i \in \{1, 2\},$$

i.e., the crossing number of a 2-page drawing does not change if we "flip" the drawing along its spine, or, in the circular model, rotate the drawing.

Finally, we group together the $2m$ elements of Types(m) that belong to a given orbit of the group, to obtain $2m \times 2m$ circulant blocks. In what follows, we denote the first row of the $2m \times 2m$ circulant block (i, j) by $q^{(i,j)} \in \mathbb{Z}^{2m}$.

LEMMA 6.1. *For odd m, the semidefinite programming bound* (14) *may be reformulated as*

$$\mathrm{SDP}_{bound}(m) = \max t$$

subject to

$$
\begin{aligned}
q_k^{(i,j)} - t - x_k^{(i,j)} &\geq 0, \quad 0 \leq k \leq 2m-1,\ 1 \leq i,j \leq 2^{m-1},\\
X_{ij}^{(t)} &= x_0^{(i,j)} + \sum_{k=1}^{2m-1} x_k^{(i,j)} e^{-\pi\sqrt{-1}tk/m}, \quad 1 \leq i \leq j \leq 2^{m-1},\ 0 \leq t \leq 2m-1,\\
X^{(t)} = (X^{(t)})^* &\succeq \mathbf{0}, \quad 0 \leq t \leq 2m-1,\\
x_k^{(i,i)} - x_{2m+1-k}^{(i,i)} &= 0, \quad 1 \leq k \leq m-1,\ 1 \leq i \leq 2^{m-1},\\
x^{(i,j)} &\in \mathbb{R}^{2m}, \quad 1 \leq i,j \leq 2^{m-1}.
\end{aligned}
$$

Proof. The proof is similar to that of Lemma 3.1 and is therefore omitted. ☐

A few remarks on the semidefinite programming reformulation in Lemma 6.1:

- As in Lemma 3.1, the constraints involve Hermitian (complex) linear matrix inequalities.
- The reduced problem has $2m$ linear matrix inequalities involving $(2^{m-1}) \times (2^{m-1})$ matrices. By comparison, the original problem had one linear matrix inequality involving a $(m2^m) \times (m2^m)$ nonnegative matrix. As a result, the reformulation in Lemma 6.1 may be solved for larger values of m than the original formulation (see next section).
- Similarly to Lemma 3.1, every feasible point $x^{(i,j)} \in \mathbb{R}^{2m}$ $(1 \leq i,j \leq 2^{m-1})$ yields a certificate of lower bound on $\text{SDP}_{bound}(m)$, and consequently a certificate of a lower bound on $\nu_2(K_{m,n})$, by (12).

7. Numerical computations: Proof of Theorem 1.2. Using the reformulation in Lemma 6.1, we numerically showed that $\text{SDP}_{bound}(7) = \frac{9}{2}$. Computation was done on a Dell Precision T7500 workstation with 92GB of RAM, using the semidefinite programming solver SDPT3 [31, 33] under MATLAB 7 together with the MATLAB package YALMIP [24]. The running time was 23,774 seconds. SDPT3 was chosen since it can deal with Hermitian matrix variables.

Using that $\text{SDP}_{bound}(7) = 9/2$, it follows from (12), (13), and (14) that

$$\nu_2(K_{7,n}) \geq (9/4)n^2 - (21/2)n. \tag{15}$$

We recall that $Z(7,n) = 9\lfloor n/2 \rfloor \lfloor (n-1)/2 \rfloor = (9/4)n^2 + O(n)$, and that $\nu_2(K_{7,n}) \leq Z(7,n)$ (since there are 2-page drawings of $K_{7,n}$ with exactly $Z(7,n)$ crossings). Using these observations and (15), Theorem 1.2 follows for $m = 7$.

Now an elementary counting argument shows that $\nu_2(K_{8,n}) \geq 8\nu_2(K_{7,n})/6$, and so using (15) and simplifying we obtain $\nu_2(K_{8,n}) \geq 3n^2 - 14n$. Since $Z(8,n) = 3n^2 + O(n)$, Theorem 1 follows for $m = 8$.

8. Concluding remarks. The Goemans–Williamson bound (section 3) empirically yields better lower bounds on $\nu_2(K_n)$ as n grows; see Figure 5.

Based on this empirical evidence, it seems reasonable to expect that the constant 0.9253 would be improved if $\mathcal{GW}(G_m)$ were computed for larger values of m. Having said that, the figure also shows a trend of diminishing returns—by extrapolating the curve in the figure, it seems that it may not be possible to improve the constant to more than 0.929, say, through computation of $\mathcal{GW}(G_m)$, if $m \leq 2{,}000$.

Another possibility to improve the constant is to compute $\nu_2(K_m)$ for larger values of m than $m = 24$, by solving the maximum cut problem in Lemma 2.1. If,

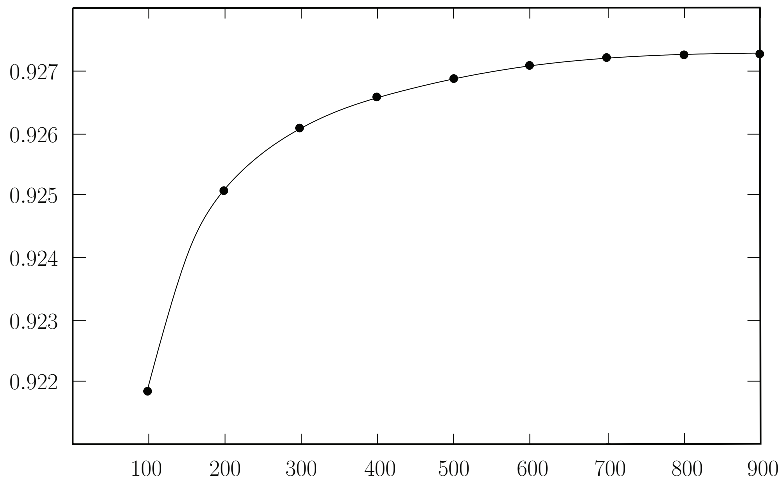

FIG. 5. *The ratio* $\frac{\binom{n}{4}-\mathcal{GW}(G_n)}{Z(n)}$ *for* $n = 99, 199, 299, 399, 499, 599, 699, 799,$ *and* 899.

for example, one could verify in this way that $\nu_2(K_{30}) = Z(30)$, then this would yield the constant 0.9297, by Claim 4.1.

Regarding the computational lower bound on $\nu_2(K_{m,n})$, it is interesting to note that the SDP bound $\mathrm{SDP}_{bound}(m)$ provided a tight asymptotic bound on $\nu_2(K_{m,n})$ for $m = 3, 5$, and 7. A similar SDP bound used in [22] and [21] did not provide a tight asymptotic bound on the usual crossing number $\mathrm{cr}(K_{m,n})$, not even for $m = 5$. Our results therefore suggest that one may be able to prove computationally that $\lim_{n\to\infty} \frac{\nu_2(K_{m,n})}{Z(m,n)} = 1$ for (fixed) odd values of $m \geq 9$. Having said that, for $m = 9$, the resulting semidefinite program was too large for us to compute $\mathrm{SDP}_{bound}(9)$. This problem therefore provides a good future challenge to the computational SDP community.

Acknowledgments. The authors are grateful to Gelasio Salazar for suggesting to work on these problems, and for providing many useful comments, suggestions, and references before deciding to withdraw from this project. The authors would also like to thank Angelika Wiegele for making the source code of her max-cut solver `BiqMac` available to them, and Imrich Vrt'o for helpful comments.

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